HOSSAM GHANEM

(11) 2.5 Continuous Functions(A)

f is continuous at $x = a$ If

$f(a)$ Exist

$f(x)$ Exist

$\lim_{x \to a} f(x) = f(a)$

Example 1 41 7 January 2012	[4 Pts.] Find all values of the constants a and b for which f is continuous at $x = -1$ $f(x) = \begin{cases} \dfrac{4b}{x-1} & \text{if } x < -1, \\ a+b & \text{if } x = -1, \\ ax^2 + x & \text{if } x > -1. \end{cases}$

Solution

$f(-1) = a + b$

$\lim_{x \to -1^-} f(x) = \frac{4b}{-1-1} = \frac{4b}{-2} = -2b$

$\lim_{x \to -1^+} f(x) = a(-1)^2 + (-1) = a - 1$

f is continuous $\quad \therefore f(-1) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$a - 1 = a + b \quad \rightarrow \quad b = -1$

$-2b = a + b \quad \rightarrow \quad 2 = a - 1 \quad \rightarrow a = 3$

Example 2 30 October 19, 2000 A	Let $f(x) = \begin{cases} \dfrac{\sqrt{x-k+1}-1}{x-k} & \text{If } x > k \\ 2x^2 & \text{If } x \leq k \end{cases}$ Find all values of k so that f is continuous on $(-\infty, \infty)$

Solution

$f(k) = 2k^2$

$\lim_{x\to k^-} f(x) = 2k^2$

$$\lim_{x\to k^+} f(x) = \lim_{x\to k^+} \frac{\sqrt{x-k+1}-1}{x-k} = \lim_{x\to k^+} \frac{(\sqrt{x-k+1}-1)(\sqrt{x-k+1}+1)}{(x-k)(\sqrt{x-k+1}+1)}$$

$$= \lim_{x\to k^+} \frac{(x-k+1)-1}{(x-k)(\sqrt{x-k+1}+1)} = \lim_{x\to k^+} \frac{(x-k)}{(x-k)(\sqrt{x-k+1}+1)}$$

$$= \lim_{x\to k^+} \frac{1}{(\sqrt{x-k+1}+1)} = \frac{1}{(\sqrt{k-k+1}+1)} = \frac{1}{2}$$

$\because$ f is cont.

$\therefore \lim_{x\to k^-} f(x) = \lim_{x\to k^+} f(x) = f(k)$

$\therefore 2k^2 = \frac{1}{2}$

$k^2 = \frac{1}{4}$

$k = \pm\frac{1}{2}$

Example 3 48 March 25, 2008 A	Let $f(x) = \begin{cases} A + \dfrac{3\lvert x-1\rvert}{x^2+x-2} & \text{If } x < 1, x \neq -2 \\ B & \text{If } x = 1 \\ \sqrt{2x-1} & \text{If } x > 1 \end{cases}$ Find the values of A and B so that f is continuous at $x = 1$

Solution

$f(1) = B$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} A + \frac{3|x-1|}{x^2+x-2} = \lim_{x\to 1^-} A + \frac{-3(x-1)}{(x+2)(x-1)} = \lim_{x\to 1^-} A - \frac{-3}{(x+2)} = A - 1$$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{2x-1} = 1$

$\therefore$ f is cont.

$f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\therefore B = 1$

$A - 1 = 1 \quad \rightarrow \quad A = 2$

Example 4
14 March 28, 1996

Let $f(x) = \begin{cases} \dfrac{\sqrt{6x-5} - \sqrt{3x-2}}{x^2+4x-5} & \text{If } x > 1 \\ \dfrac{A}{4} & \text{If } x = 1 \\ \dfrac{5|x-1|}{x^2-3x+2} + B & \text{If } x < 1 \end{cases}$

Find the values of A and B so that f is continuous for every $x \in (-\infty, \infty)$

Solution

$$f(1) = \frac{1}{4}A$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{5|x-1|}{x^2-3x+2} + B = \lim_{x\to1^-} \frac{-5(x-1)}{(x-2)(x-1)} + B = \lim_{x\to1^-} \frac{-5}{(x-2)} + B = B+1$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{\sqrt{6x-5} - \sqrt{3x-2}}{x^2+4x-5} = \lim_{x\to1^+} \frac{\left(\sqrt{6x-5} - \sqrt{3x-2}\right)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)}{(x^2+4x-5)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)}$$

$$= \lim_{x\to1^+} \frac{(6x-5) - (3x-2)}{(x+5)(x-1)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)} = \lim_{x\to1^+} \frac{6x-5-3x+2}{(x+5)(x-1)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)}$$

$$= \lim_{x\to1^+} \frac{3x-3}{(x+5)(x-1)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)} = \lim_{x\to1^+} \frac{3(x-1)}{(x+5)(x-1)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)}$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{3}{(x+5)\left(\sqrt{6x-5} + \sqrt{3x-2}\right)} = \frac{3}{6(1+1)} = \frac{1}{4}$$

$\because f$ is cont.

$$f(1) = \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$$

$$\therefore \frac{1}{4}A = \frac{1}{4} \quad \rightarrow \quad A = 1$$

$$B+1 = \frac{1}{4} \quad \rightarrow \quad B = -\frac{3}{4}$$

Example 5 33 January 20, 2009 A	Let $f(x)=\begin{cases}\dfrac{x^2-k}{x^2+1}, & \text{if } x\ge 0,\\ \dfrac{x^3-k+1}{x^2+2}, & \text{if } x<0.\end{cases}$ (a) Find the value of k such that f is continuous at $x=0$. (b) Is f continuous at $x=3$? Justify your answer.

Solution

(a)

$$f(0)=\frac{0-k}{0+1}=-k$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x^2-k}{x^2+1}=-k$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x^3-k+1}{x^2+2}=\frac{-k+1}{2}$$

$\therefore$ f is cont. at $x=0$

$$f(0)=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$$

$\therefore \frac{-k+1}{2}=-k$ $\rightarrow$ $\therefore -k+1=-2k$ $\rightarrow$ $k=-1$

(b)

$$f(x)=\frac{x^2-1}{x^2+1}\quad ,\quad x\ge 0$$

$$f(3)=\frac{9-1}{9+1}=\frac{8}{10}=\frac{4}{5}$$

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^2-1}{x^2+1}=\frac{9-1}{9+1}=\frac{8}{10}=\frac{4}{5}$$

$\therefore f(3)=\lim_{x\to 3} f(x)$

$\therefore$ f is cont. at $x=3$

Example 6 31 June 5, 2008	Let $f(x)=\dfrac{x^2+x}{\lvert x\rvert}$, Where $x\ne 0$. Can f be defined at $x=0$ So that , f becomes continuous ? Justify your answer.

Solution

$$f(x)=\frac{x^2+x}{|x|}$$

if $x>0$

$$f(x)=\frac{x^2+x}{x}=\frac{x(x+1)}{x}=x+1$$

if $x<0$

$$f(x)=\frac{x^2+x}{-x}=\frac{x(x+1)}{-x}=-(x+1)$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} -(x+1)=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x+1=1$$

$\therefore \lim_{x\to 0} f(x)$ $D.N.E$

$\therefore$ f is discont. at $x=0$

Homework

1 12 November 2, 1995	Let $f(x)=\begin{cases}\dfrac{x^2-1}{\sqrt{x}-1} & \text{If } x\neq 1\\ K & \text{If } x=1\end{cases}$ Find the value of K for which $f(x)$ is continuous at $x=1$
2 7 July 29, 1993	Let $f(x)=\begin{cases}\dfrac{\sqrt{6x-5}-\sqrt{3x+10}}{x-5} & \text{If } x>5\\ \dfrac{A}{10} & \text{If } x=5\\ \dfrac{7\lvert x-5\rvert}{x^2-3x-10}+B & \text{If } x<5\end{cases}$ Find A and B so that f is continuous at $x=5$
3	The function f is not defined at $x=0$. Define $f(x)$ so that f is continuous for $x=0$ which $f(x)=\dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$
4	Let $f(x)=\begin{cases}x^2+M & \text{If } x\geq 0\\ 4x+N & \text{If } x<0\end{cases}$ and $f(1)=5$ Determine the values of M and N so that $f(x)$ is continuous on $\mathcal{R}$
5	The function f is not defined at $x=0$. Define $f(x)$ so that f is continuous for $x=0$ which $f(x)=\dfrac{(1+x^2)-1}{x}$
6	Let $f(x)=\begin{cases}\dfrac{x^2-9}{\lvert x-3\rvert} & \text{If } x<3\\ b & \text{If } x=3\\ ax & \text{If } x>3\end{cases}$ Find a and b so that f is continuous at x = 3

Homework

7 37 June 6, 2010	Find all intervals on which the function $f(x)=\dfrac{\sqrt{9-x^2}}{x^4-16}$ is continuous.
8 39 5 June, 2011	[4 pts.] Let $f(x)=\begin{cases} ax+b & \text{for } x\le 0 \\ x^2+a-b & \text{for } 0<x\le 2 \\ \cos(x-2) & \text{for } x>2. \end{cases}$ Find the values of a and b for which f is continuous on $(-\infty,\infty)$
9 2 November 9, 1989	Suppose that $f(x)=\begin{cases} 4x & \text{If } x<-1 \\ ax+b & \text{If } -1\le x\le 2 \\ -5x & \text{If } x>0 \end{cases}$ Find the values of a and b such that $f(x)$ is continuous at -1 and 2
10 4 May 19, 1992	Let $f(x)=\begin{cases} x^2+2 & \text{If } x\le 0 \\ Ax+B & \text{If } 0<x\le 3 \\ \dfrac{x^2-9}{x-3} & \text{If } x>3 \end{cases}$ Find the constants A and B such that f is continuous for all real numbers

9 2 November 9, 1989	Suppose that $f(x) = \begin{cases} 4x & \text{If} \quad x < -1 \\ ax + b & \text{If} \quad -1 \le x \le 2 \\ -5x & \text{If} \quad x > 0 \end{cases}$ Find the values of a and b such that $f(x)$ is continuous at -1 and 2

Solution

at $x = 1$

$f(-1) = -a + b$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 4x = -4$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax + b = -a + b$

$\therefore b - a = -4 \quad \rightarrow\rightarrow \quad a - b = 4$

at $x = 2$

$f(2) = 2a + b$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x + b = 2a + b$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -5x = -10$

$\therefore 2a + b = -10 \quad \rightarrow \boxed{1}$

$a - b = 4 \quad \rightarrow \boxed{2}$

$\boxed{1} + \boxed{2}$

$3a = -6 \quad \rightarrow \quad a = -2$

$b - a = -4$

$b + 2 = -4 \quad \rightarrow \quad b = -6$

10 4 May 19, 1992	Let $f(x) = \begin{cases} x^2 + 2 & \text{If} \quad x \le 0 \\ Ax + B & \text{If} \quad 0 < x \le 3 \\ \dfrac{x^2 - 9}{x - 3} & \text{If} \quad x > 3 \end{cases}$ Find the constants A and B such that f is continuous for all real numbers

Solution

$f(0) = 2$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 + 2 = 2$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} Ax + B = B$

$\because f$ is cont.

$f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\therefore B = 2$

$f(3) = A(3) + 2 = 3A + 2$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} Ax + B = 3A + 2$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \dfrac{x^2 - 9}{x - 3} = \lim_{x\to 3^+} \dfrac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3^+} x + 3 = 6$

$\because f$ is cont.

$f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$\therefore 3A + 2 = 6 \quad \rightarrow \quad 3A = 4 \quad \rightarrow \quad A = \dfrac{4}{3}$